

On the Hilbert Transform and Lacunary Directions in the Plane

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Abstract

We show that the maximal operator below, defined initially for Schwartz functions f on the plane, extends to a bounded operator from $L^p(\mathbb{R}^2)$ into itself for $1 < p < \infty$.

$$\sup_{k \in \mathbb{Z}} \left| \text{p.v.} \int f(x - (1, 2^k)y) \frac{dy}{y} \right|.$$

1 Introduction

For a smooth rapidly decreasing function f on the plane and a nonzero vector $v \in \mathbb{R}^2$, define

$$H_v f(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x - yv) \frac{dy}{y},$$

which is the one dimensional Hilbert transform of f computed in the direction v . This definition is independent of the length of v . For a countable collection of vectors $V \subset \mathbb{R}^2$, we define a maximal function

$$H^V f(x) := \sup_{v \in V} |H_v f(x)|.$$

We shall prove

1.1. Theorem. *Suppose $V = \{(1, a_k) : k \in \mathbb{Z}\}$ is such that there is a $\lambda > 1$ so that for all k*

$$0 < a_{k+1} < a_k/\lambda.$$

Then the operator H^V extends to a bounded operator from $L^p(\mathbb{R}^2)$ to itself for all $1 < p < \infty$.

The theory of the Hilbert transform and maximal function of one variable are closely intertwined. And the maximal function version of the theorem above was proved in a series of papers [9, 2], first in the L^2 case and last of all for all L^p , $1 < p < \infty$, [8]. But the method of proof employed does not seem to imply the theorem above.

In a related matter, one can consider bounds on H^V which depend only on the cardinality of V . On $L^2(\mathbb{R}^2)$, the bound of $\log \#V$ follows more or less immediately from the Rademacher-Menshov theorem. But maximal function variants were only recently established by N. Katz [4, 5], using a subtle range of ideas.

We shall prove the theorem above by invoking the BMO theory of the bidisk, as developed by S.Y. Chang and R. Fefferman [1]. This is conveniently done via a combinatorial model of H^V , and once it is in place, a maximal inequality can be established by way of an argument nearly devoid of the geometry of the plane, as all the relevant geometric facts are already encoded into the BMO theory.

Indeed, the salient features of our argument are (1) a proper notion of “energy” arising directly from a Bessel inequality, as used in e.g. [7], (2) a closely related notion of “charge” and (3) a John–Nirenberg inequality. Very little else is needed to conclude a maximal inequality, making the proof adaptable to other situations with the same attributes. This observation bears some resemblance to the method of approach in N. Katz’ approach to the maximal function in arbitrary directions, [5].

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2 The Combinatorial Model

Define the Fourier transform on \mathbb{R} as $\hat{f}(\xi) = \int e^{-ix\xi} f(x) dx$. We will use the same notation and a similar definition for the Fourier transform on the plane \mathbb{R}^2 . Set $\langle f, g \rangle := \int f \bar{g} dx$. And by $A \lesssim B$, we mean that for some absolute constant K , $A \leq KB$.

We will replace the Hilbert transform by $Pf(x) = \int_0^\infty e^{ix\xi} \hat{f}(\xi) d\xi$, which is Fourier projection onto the positive frequencies of f . P is a linear combination of the identity and the Hilbert transform. Hence, in complete analogy to the definition of H_v , we can define $P_v f$ as the one one-dimensional transform applied in the direction v to the function f defined on the plane.

It suffices to consider maximal functions constructed from P_v , and more particularly, it suffices to consider collections of vectors $V = \{(-1, a_k) : k \in \mathbb{N}\}$ with $1/2 < 2^k a_k < 1$ for $k \in \mathbb{N}$. Throughout the rest of this section we consider such collections V , and we define P^V in complete analogy to the definition of H^V .

On the plane, view points as $x = (x_1, x_2)$, with the dual frequency variables being (ξ_1, ξ_2) . We define

$$Bf(x) = \int_0^\infty \int_0^\infty \hat{f}(\xi_1, \xi_2) e^{ix \cdot (\xi_1, \xi_2)} d\xi_1 d\xi_2,$$

which is Fourier projection onto $[0, \infty)^2$. It suffices to consider the maximal function $P^V Bf$. Our purpose right now is to write B as a limit of two different combinatorial sums. This will permit a corresponding decomposition of the maximal operator.

For $j = 1, 2$, consider Schwartz functions φ^j on \mathbb{R} such that

$$\begin{aligned} \mathbf{1}_{[\frac{7}{8}, \frac{13}{8}]}(\xi) &\leq \widehat{\varphi^1}(\xi) \leq \mathbf{1}_{[\frac{3}{4}, \frac{7}{4}]}(\xi), \\ \widehat{\varphi^2}(\xi) &> 0, \quad \xi > 0, \\ |\widehat{\varphi^2}(\xi)| &\leq \min\{|\xi|, |\xi|^{-1}\}, \\ \varphi^2(x) &\text{ is compactly supported.} \end{aligned}$$

Given a rectangle $R = r_1 \times r_2$ in the plane, set

$$\varphi_R^j(x_1, x_2) = \prod_{k=1}^2 |r_k|^{-1/2} \varphi^k\left(\frac{x_k - c(r_k)}{|r_k|}\right).$$

Note that the L^2 norm of this function is independent of the choice of R .

We shall consider classes of rectangles specified by $\lambda = (\lambda_1, \lambda_2) \in [1, 2]^2$ and $y \in \mathbb{R}^2$. Set

$$\mathcal{R}_{\lambda, y} = \left\{ y + \prod_{j=1}^2 [\lambda_j m_j 2^{n_j}, \lambda_j (m_j + 1) 2^{n_j}) : (m_1, m_2), (n_1, n_2) \in \mathbb{Z}^2 \right\}.$$

We write $\mathcal{R}_{(1,1),(0,0)}$ as \mathcal{R} , which is just (one choice of) all dyadic rectangles in the plane.

Notice that the functions $\{\varphi_R : R \in \mathcal{R}_{\lambda, y}\}$ are obtained from $\{\varphi_R : R \in \mathcal{R}\}$ by dilating x_1 by a factor of λ_1 , x_2 by a factor of λ_2 with both dilations preserving the L^2 norm and then translating by y . Also note that there is no need to consider dilations by factors greater than 2, since a dyadic grid on \mathbb{R} is invariant under dilations by 2.

Define two operations by sums over these collections of rectangles.

$$C_{\lambda, y}^j f(x) := \sum_{R \in \mathcal{R}_{\lambda, y}} \langle f, \varphi_R^1 \rangle \varphi_R^j(x), \quad j = 1, 2.$$

We set $C^j := C_{(1,1),(0,0)}^j$.

We use these to build two limiting representations of B , and to this end we note that B is characterized, up to a constant multiple, as a non-zero linear operator on $L^2(\mathbb{R}^2)$ that is (1) translation invariant (2) invariant under dilations of both variables independently, and (3) $Bf = 0$ if \hat{f} is not supported on $[0, \infty)^2$. (This is suggested by the decomposition in second section in [7].)

Define for $j = 1, 2$,

$$B^j f(x) = \lim_{Y \rightarrow \infty} \int \int_{D(Y)} C_{(2^{\lambda_1}, 2^{\lambda_2}), y}^j f \mu(d\lambda_1, d\lambda_2, dy),$$

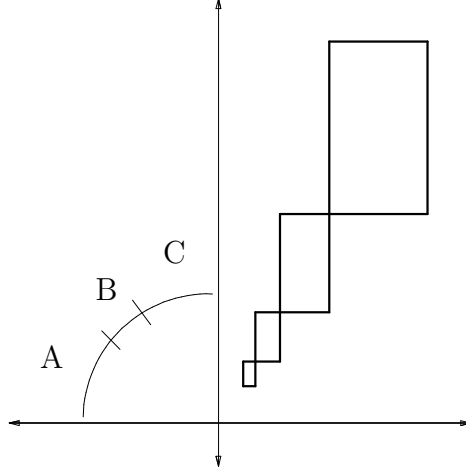


Figure 1: On the right are some rectangles R in a collection $\mathcal{R}(k)$. If a vector v is in arc A , then $P_v \varphi_R^1 = 0$, whereas if v is in arc C , then $P_v \varphi_R^1 = \varphi_R^1$. But if v is in arc B , then $P_v \varphi_R^1$ need not be 0 nor φ_R^1 .

where μ is normalized Lebesgue measure on $D(Y) := [1, 2)^2 \times \{y : |y| < Y\}$. [Note that we are averaging with respect to multiplicative Haar measure in the dilation parameters λ_1 and λ_2 .] The limit is seen to exist for smooth compactly supported functions. The operators B^j extend to bounded linear operators on L^2 . They are translation and dilation invariant since we average over all possible dilations and translations. Moreover, $B^j f = 0$ if \hat{f} is not supported on $[0, \infty)^2$. It remains to show that B^j , $j = 1, 2$ are non-zero operators and so are constant multiples of B . Indeed, B^1 is easily seen to be positive semidefinite, by the choice of φ^1 .

For B^2 , observe that if we set $\tau_y \varphi(x) = \varphi(x - y)$ for functions on \mathbb{R} , then we have

$$\begin{aligned} \int_0^1 \sum_{n=-\infty}^{\infty} \langle f, \tau_{n+y} \varphi^1 \rangle \tau_{n+y} \varphi^2(x) dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) \overline{\varphi^1(-z)} \varphi^2(x - z) dy dz \\ &= f * \psi(x) \end{aligned}$$

where $\psi(x) = \int_{-\infty}^{\infty} \overline{\varphi^1(y)} \varphi^2(x + y) dy$. Note that $\hat{\psi}(\xi) = \overline{\hat{\varphi}^1(\xi)} \hat{\varphi}^2(\xi)$. A direct computation now shows that B^2 is non-zero.

We conclude that C^j are constant multiples of B . Hence, to prove our theorem it suffices to prove a bound for

$$\sup_{v \in V} |C^2 P_v C_{\lambda, y}^1 f|, \quad \lambda \in [1, 2)^2, \quad y \in \mathbb{R}^2.$$

We demonstrate a bound that is uniform in λ and y . Then we can average these inequalities to conclude the same for $\sup_{v \in V} |B^2 P_v B^1|$, which is sufficient.

Upon expansion of the term $C^2 P_v C_{\lambda, y}^1 f$ we obtain the inner product $\langle P_v \varphi_R^1, \varphi_{R'}^1 \rangle$ for $R \in \mathcal{R}$ and $R' \in \mathcal{R}_{\lambda, y}$. This inner product is at most one in modulus. Moreover, recall that we consider v of the form $(-1, a)$, and write $R = r_1 \times r_2$. Then

$$P_{(-1, a)} \varphi_R^1 = \begin{cases} \varphi_R^1 & \frac{a}{2} |r_2| > |r_1| \\ 0 & |r_1| > 2a |r_2| \end{cases}$$

Thus, it is natural to two cases, the first being the sum taken over those rectangles with $P_v \varphi_R^1 = \varphi_R^1$ and the second is over those rectangles with $P_v \varphi_R^1 \notin \{0, \varphi_R^1\}$. See Figure 1.

The second case concerns classes of rectangles which we define this way. Recall that $V = \{(-1, a_k) : k \in \mathbb{N}\}$ and set

$$\mathcal{R}_{\lambda, y}(k) := \{r_1 \times r_2 \in \mathcal{R}_{\lambda, y} : 2a_k |r_2| \geq |r_1| \geq \frac{a_k}{2} |r_2|\}.$$

We again set $\mathcal{R}(k) := \mathcal{R}_{(1,1),(0,0)}(k)$. These are the rectangles for which $P_{(-1,a_k)}\varphi_R^1$ need not be 0 or φ_R^1 . We define

$$\Phi_{\lambda,y,k}^j f := \sum_{r \in \mathcal{R}_{\lambda,y}} \langle f, \varphi_R^1 \rangle \varphi_R^j, \quad j = 1, 2,$$

and set $\Phi_k^j := \Phi_{(1,1),(0,0),k}^j$. Then, in the case of $1 < p < 2$, the term to bound is

$$\begin{aligned} \|\sup_k |\Phi_{\lambda,y,k}^2 P_{(-1,a_k)} \Phi_k^1 f|\|_p &\leq \left\| \left[\sum_k |\Phi_{\lambda,y,k}^2 P_{(-1,a_k)} \Phi_k^1 f|^2 \right]^{1/2} \right\|_p \\ &\lesssim \left\| \left[\sum_k |P_{(-1,a_k)} \Phi_k^1 f|^2 \right]^{1/2} \right\|_p \\ &\lesssim \|f\|_p \end{aligned}$$

The penultimate line is a vector valued Calderon-Zygmund inequality and the last line follows from lemma 4.1 below. The case of $2 \leq p < \infty$ is even easier.

In the first case, there is the important point that the inner product $\langle \varphi_R^1, \varphi_{R'}^1 \rangle$ will be zero unless the side lengths of R and R' agree. And if they do, the inner product will decay as a function of the relative distance between $R \in \mathcal{R}$ and $R' \in \mathcal{R}_{\lambda,y}$. This relative distance will in addition be influenced by y , indeed, if $|y|$ is considerably greater than the side lengths of both rectangles, then it is the dominant term in determining the relative distances between the two rectangles.

Thus, in seeking a useful quantitative estimate, it is useful to link the translation parameter y to the scales of the rectangles involved. This we shall explicitly do in this definition. The maximal operator we control is

$$(2.1) \quad \sup_v \left| \sum_{R \in \mathcal{R}} \langle f, \varphi_R^1 \rangle \langle P_v \varphi_R^1, \varphi_{\sigma(R)}^1 \rangle \varphi_{\sigma(R)}^2 \right|.$$

In this display, the map σ is given by

$$(2.2) \quad \sigma(r_1 \times r_2) = \lambda_1 r_1 \times \lambda_2 r_2 + (y_1 |r_1|, y_2 |r_2|) + (\delta_1(r_1), \delta_2(r_2)),$$

in which $(\lambda_1, \lambda_2) \in (1, 2]^2$ is fixed, $y = (y_1, y_2) \in \mathbb{R}^2$ is fixed and $\delta_j(\cdot)$, $j = 1, 2$ are two functions from the dyadic intervals on \mathbb{R} into \mathbb{R}_+ , with $0 \leq \delta_j(r) \leq |r|$ for all dyadic intervals r .

Our purpose is to prove a bound on the L^p norm of the operator in (2.1) which decays rapidly in $|y|$. This is possible because of the estimate

$$|\langle \varphi_R^1, \varphi_{\sigma(R)}^1 \rangle| \leq C(1 + |y_1| + |y_2|)^{-10}.$$

Therefore, the control of this part of the supremum will follow from the lemma of the next section. This completes our proof of the maximal theorem.

3 A Discrete Maximal Inequality

For rectangles $R = r_1 \times r_2 \in \mathcal{R}$ we set $\text{sl}(R) := |r_1|/|r_2|$.

3.1. Lemma. *Let $y = (y_1, y_2) \in \mathbb{R}^2$ and let σ be as in (2.2). For all $1 < p < \infty$ for arbitrary choices of signs $\{\varepsilon_R : R \in \mathcal{R}\}$, with $\varepsilon_R \in \{\pm 1\}$, the maximal operator below maps $L^p(\mathbb{R}^2)$ into itself.*

$$\sup_{s>0} \left| \sum_{\substack{R \in \mathcal{R} \\ \text{sl}(R) \geq s}} \varepsilon_R \langle f, \varphi_R^1 \rangle \varphi_{\sigma(R)}^2 \right|$$

The norm of the operator is at most $C_p(|y_1| + |y_2|)^{3/p}$.

Let \mathcal{S} denote an arbitrary finite subset of \mathcal{R} and set

$$A_{\max}^{\mathcal{S}} = \sup_{a>0} \left| \sum_{\substack{R \in \mathcal{S} \\ \text{sl}(R) > a}} \varepsilon_R \langle f, \varphi_R^1 \rangle \varphi_{\sigma(R)}^2 \right|$$

where $f \in L^p(\mathbb{R}^2)$ is a fixed function of norm one. By $A^{\mathcal{S}}$ denotes the same sum over $R \in \mathcal{S}$, without the supremum over a . It suffices to show that there is a constant K_p independent of f and $\mathcal{S} \subset \mathcal{R}$, for which

$$|\{A_{\max}^{\mathcal{S}} > 1\}| \lesssim (|y_1| + |y_2|)^3.$$

It follows that $A_{\max}^{\mathcal{S}}$ maps L^p into weak L^p with norm at most a constant times $(|y_1| + |y_2|)^{3/p}$. Interpolation then proves the Lemma.

We define the shadow of \mathcal{S} to be $\text{sh}(\mathcal{S}) = \bigcup_{R \in \mathcal{S}} R$. Since we specified that φ^2 has compact support, it follows that

$$(3.2) \quad \text{supp}(A^{\mathcal{S}}) \subset \{M\mathbf{1}_{\text{sh}(\mathcal{S})} \geq \delta(1 + |y_1| + |y_2|)^{-2}\}$$

for an absolute choice of $\delta > 0$, where M is the strong maximal function. Thus, M can be defined as

$$Mg(x) := \sup_{\substack{R \in \mathcal{R} \\ x \in R}} |2R|^{-1} \int_{2R} |g(y)| dy.$$

Another definition we need is the “energy of a collection of rectangles \mathcal{S} ”

$$\text{eng}(\mathcal{S}) := \sup_{\mathcal{S}' \subset \mathcal{S}} |\text{sh}(\mathcal{S}')|^{-1} \left\| \left[\sum_{R \in \mathcal{S}'} \left| \frac{\langle f, \varphi_R^1 \rangle}{\sqrt{|R|}} \right|^2 \mathbf{1}_R \right]^{1/2} \right\|_1$$

This quantity is related to the definition of BMO of the bidisk, as we have the equivalence $\text{eng}(\mathcal{R}) \simeq \|f\|_{\text{BMO}}$. See [1, 3]. We shall specifically need the fact that

$$\left\| \left[\sum_{R \in \mathcal{S}} \left| \frac{\langle f, \varphi_R^1 \rangle}{\sqrt{|R|}} \right|^2 \mathbf{1}_R \right]^{1/2} \right\|_p \lesssim \text{eng}(\mathcal{S}) |\text{sh}(\mathcal{S})|^{1/p}.$$

This is a manifestation of the John–Nirenberg inequality for the BMO space. See [1], or the concluding section of this paper.

Say that \mathcal{S} has charge δ if $\text{eng}(\mathcal{S}) \leq \delta$ and

$$\sum_{R \in \mathcal{S}} |\langle f, \varphi_R^1 \rangle|^2 \geq \frac{\delta^2}{4} |\text{sh}(\mathcal{S})|.$$

There are two essential aspects of this definition. For the first, if \mathcal{S} has charge δ then we may apply the Littlewood–Paley inequality, yielding

$$\delta |\text{sh}(\mathcal{S})|^{1/p} \lesssim \left\| \left[\sum_{R \in \mathcal{S}} \left| \frac{\langle f, \varphi_R^1 \rangle}{\sqrt{|R|}} \right|^2 \mathbf{1}_R \right]^{1/2} \right\|_p \lesssim \|f\|_p$$

The second fact concerns a collection of rectangles \mathcal{S} of energy δ . It necessarily contains a subset of charge δ . Suppose there are two disjoint subsets \mathcal{S}_1 and \mathcal{S}_2 of \mathcal{S} , of charge δ . Then $\mathcal{S}_1 \cup \mathcal{S}_2$ also has charge δ . Therefore, \mathcal{S} contains a (non-unique) maximal subcollection of charge δ .

Having finished with definitions, the main argument begins. If we begin with a finite collection of rectangles \mathcal{S} , it has finite energy. We now reduce to the case in which $\text{eng}(\mathcal{S})$ is at most one. Indeed, \mathcal{S} is the union of collections \mathcal{S}_j for $j \geq 0$ with the energy of \mathcal{S}_0 being at most one, and the charge of \mathcal{S}_j being 2^j for all $j > 0$. As we are to prove a distributional inequality, it follows, after an application of (3.2), that we need not consider those collections \mathcal{S}_j with $j \geq 0$.

Thus we can assume that the energy of \mathcal{S} is at most one. We shall show that for any $q > p$,

$$\|A_{\max}^{\mathcal{S}}\|_q \leq K_q \text{eng}(\mathcal{S})^{1-2p/q}.$$

That is, the estimate is now independent of y_1 and y_2 and depends on $p > 1$ only through the implied constant in the inequality. This will conclude the proof of our lemma.

The construction which leads to this inequality begins now. Define for integers j

$$\mathcal{S}(j) := \{R \in \mathcal{S} : \text{sl}(R) \geq 2^j\}.$$

Let K_v , for integers $v \geq 0$, be subsets of \mathbb{Z} such that

- $K_0 = \{j_0\}$ for some integer with $\mathcal{S}(j_0) = \emptyset$.
- $K_v \subset K_{v+1}$ for all v .
- For all $j \in \mathbb{Z}$ if $j_v \in K_v$ is the maximal element of K_v less than or equal to j then

$$\text{eng}(\mathcal{S}(j) - \mathcal{S}(j_v)) \leq 2^{-v}.$$

- The cardinality of K_v is minimal, subject to the first two conditions.

Notice that for each j and v with $j_v \neq j_{v-1}$, we have that $\text{eng}(\mathcal{S}(j_v) - \mathcal{S}(j_{v-1})) \geq 2^{-v-1}$. Then, we have for any integer j ,

$$|A^{\mathcal{S}(j)}| \leq \sum_{v=1}^{\infty} |A^{\mathcal{S}(j_v)} - A^{\mathcal{S}(j_{v-1})}|$$

And so it suffices to prove the bound

$$\left\| \sup_j |A^{\mathcal{S}(j_v)} - A^{\mathcal{S}(j_{v-1})}| \right\|_q \lesssim 2^{-v(1-p/q)}, \quad q > 2p.$$

For this last inequality, we again apply the decomposition of a set of rectangles into subsets with charge. Namely, for integers $w \geq v$ there are collections \mathbb{S}_w of subsets of \mathcal{S} such that

- For all w , every $\mathcal{S}' \in \mathbb{S}_w$ has charge 2^{-w} .
- For all w , the collections $\mathcal{S}' \in \mathbb{S}_w$ are pairwise disjoint.
- For each v and $w \geq v$, there is an $\mathcal{S}'(v, w) \in \mathbb{S}_w$ such that

$$\mathcal{S}(j_v) - \mathcal{S}(j_{v-1}) = \bigcup_{w=v}^{\infty} \mathcal{S}'(v, w).$$

This is achieved just by applying, to each collection of rectangles $\mathcal{S}(j_v) - \mathcal{S}(j_{v-1})$, the first decomposition above, adding in an additional step of pulling out maximal subcollections of appropriate charge.

The collection \mathbb{S}_w possesses the following property.

$$\sum_{\mathcal{S}' \in \mathbb{S}_w} |\text{sh}(\mathcal{S}')| \lesssim 2^{\max(2, p)w}.$$

Indeed, set $\mathcal{S}^w := \bigcup_{\mathcal{S}' \in \mathbb{S}_w} \mathcal{S}'$, which is a collection of rectangles of energy at most one, by the first step of our construction. In addition, set

$$g := \sum_{R \in \mathcal{S}^w} \langle f, \varphi_r^1 \rangle \varphi_r^2.$$

This function is supported on $\text{sh}(\mathcal{S}^w)$. And, as each $\mathcal{S}' \in \mathbb{S}_w$ has charge 2^{-w} ,

$$\sum_{\mathcal{S}' \in \mathbb{S}_w} |\text{sh}(\mathcal{S}')| \lesssim 2^{2w} \|g\|_2^2.$$

And so we estimate the L^2 norm of g .

At this point, we consider the case of $2 \leq p$. In this case, we have in addition $\|g\|_p \lesssim \|f\|_p \lesssim 1$, so that $\|g\|_2 \lesssim |\text{sh}(\mathcal{S}^w)|^{\frac{1}{2} - \frac{1}{p}} \|g\|_p$. And this proves our claim.

Now, if $1 < p < 2$, then observe that $\|g\|_{BMO} \lesssim 1$, so that

$$\|g\|_2 \lesssim \|g\|_p^{p/2} \|g\|_{BMO}^{1-p/2} \lesssim 1.$$

And this finishes the proof of our observation.

To conclude, we may estimate

$$\begin{aligned} \left\| \sup_{\mathcal{S}' \in \mathbb{S}_w} |A^{\mathcal{S}'}| \right\|_q^q &\leq \sum_{\mathcal{S}' \in \mathbb{S}_w} \|A^{\mathcal{S}'}\|_q^q \\ &\lesssim 2^{-qw} \sum_{\mathcal{S}' \in \mathbb{S}_w} |\text{sh}(\mathcal{S}')| \\ &\lesssim 2^{-(q-2p)w}. \end{aligned}$$

The proof of this lemma is done, as the q th root of the last estimate is summable in $w \geq v$, provided $q > 2p$.

4 The Diagonal Terms

We prove the lemma

4.1. Lemma. *For $1 < p < \infty$, we have the inequality*

$$\left\| \left[\sum_k |P_{(-1, a_k)} \Phi_k^1 f|^{p^*} \right]^{1/p^*} \right\|_p \lesssim \|f\|_p, \quad p^* := \max(2, p).$$

The proof depends heavily on inequalities of Littlewood–Paley type, namely

$$\|f\|_p \simeq \left\| \left[\sum_k |\Phi_k^j f|^2 \right]^{1/2} \right\|_p \simeq \left\| \left[\sum_{R \in \mathcal{R}} \frac{|\langle f, \varphi_R^j \rangle|^2}{|R|} \mathbf{1}_R \right]^{1/2} \right\|_p,$$

where $1 < p < \infty$ and $j = 1, 2$. These are obtained from applications of the ordinary Littlewood–Paley inequalities in each variable separately.

The proof of the Lemma for $p \geq 2$ is now at hand.

$$\begin{aligned} \sum_k \|P_{(-1, a_k)} \Phi_k^1 f\|_p^p &\lesssim \sum_k \|\Phi_k^1 f\|_p^p \\ &\lesssim \left\| \left[\sum_k |\Phi_k^1 f|^2 \right]^{1/2} \right\|_p^p \\ &\lesssim \|f\|_p^p. \end{aligned}$$

But for the case of $1 < p < 2$, we rely upon a more substantive approach. We in fact show that the operators

$$T^* f := \sum_{k=1}^{\infty} \varepsilon_k P_{(-1, a_k)} \Phi_k^1 f, \quad \varepsilon_k \in \{\pm 1\},$$

map L^p into itself for $1 < p < 2$. The constants involved are shown to be independent of the choices of signs. From this, the lemma follows.

We use duality and prove the corresponding fact on the dual operator T . One readily checks that T is bounded on L^2 and the point is to extend this fact to L^p for $p > 2$. Moreover, by the Littlewood–Paley inequalities, it suffices to prove the square function bound we state now. Set

$$\phi_R := P_{(-1, a_k)} \varphi_R^1, \quad R \in \mathcal{R}(k), \quad k \in \mathbb{N}.$$

Then we shall establish

$$(4.2) \quad \left\| \left[\sum_{R \in \mathcal{R}} \frac{|\langle f, \phi_R \rangle|^2}{|R|} \mathbf{1}_R \right]^{1/2} \right\|_p \lesssim \|f\|_p, \quad 2 < p < \infty.$$

This is clear for $p = 2$ and so we see a second endpoint estimate with which to interpolate. The endpoint estimate is that the square function maps L^∞ into BMO of the bidisk. This fact requires that we prove the inequality

$$(4.3) \quad \sum_{R \subset U} |\langle f, \phi_R \rangle|^2 \lesssim |U| \|f\|_\infty^2,$$

for all open sets $U \subset \mathbb{R}^2$ and functions f .

Once this is established, one can interpolate to deduce (4.2). In fact the BMO estimate directly supplies the restricted weak-type version of (4.2). [The notions of energy and charge are relevant to this argument. The details are left to the reader.] Then standard interpolation supplies the inequality (4.2).

Our proof of (4.3) follows routine lines of argument, for the BMO theory of the bidisk. We fix an open set U and function f bounded by 1. By the evident L^2 estimate, it suffices to consider the case in which f is supported off of the set $\{M \mathbf{1}_U > \frac{1}{2}\}$. For a dyadic rectangle $R \subset U$ set

$$\mu_R := \sup\{\mu > 0 : \mu R \subset \{M \mathbf{1}_U > \frac{1}{2}\}\}.$$

This quantity is at least 2. To a rectangle $R = r_1 \times r_2$ we associate a set of dyadic rectangles

$$\mathcal{S}(R, j, \ell) := \{R' = r'_1 \times r'_2 \in \mathcal{R} : R' \subset R, r_j = r'_j, |R'| = 2^{-\ell}|R|\}, \quad j = 1, 2, \ell = 1, 2, \dots$$

Set $\mathcal{S}(R) = \bigcup_{j=1}^2 \bigcup_{\ell=1}^{\infty} \mathcal{S}(R, j, \ell)$. The principle fact to prove is that for each $R \subset U$,

$$(4.4) \quad \sum_{R' \subset \mathcal{S}(R)} |\langle \phi_{R'}, f \rangle|^2 \lesssim \mu_R^{-1/20} |R|.$$

This is so under the additional assumption that f is bounded by 1 and supported off of the set $\{M\mathbf{1}_U > \frac{1}{2}\}$.

That this proves (4.3) follows from an application of Journé's Lemma, [6]. The details are left to the reader. [Virtually the only way to verify the Carleson measure condition for a specific measure is through this basic lemma of Journé.]

There is a further reduction in (4.4) to make. We assume that $\mu > 2$, that f is bounded by 1 and supported on $2\mu R - \mu R$. Then we show that

$$(4.5) \quad \sum_{R' \subset \mathcal{S}(R)} |\langle \phi_{R'}, f \rangle|^2 \lesssim \mu^{-1/8} |R|.$$

This proves our desired inequality, as is easy to see. This inequality depends upon specific properties of the transformation P_v .

Now, for $R' \in \mathcal{R}(k)$, the function $\phi_{R'} = P_{(-1, a_k)} \varphi_{R'}^2$ need not be 0 or $\varphi_{R'}^2$, and we have the following estimate.

$$(4.6) \quad |\phi_{R'}(x)| \lesssim |R'|^{-1/2} \{1 + |r'_1| |[(x - c(R')) \cdot (-1, a_k)] + (|r'_2| |[(x - c(R')) \cdot (a_k, 1)])^{200}\}^{-1}$$

A standard calculation verifies this, recalling that the kernel associated to P —and hence P_v —has decay of order $1/y$. It then follows that if in addition $R' \in \mathcal{S}(R, j, \ell)$, we then have

$$\int_{2\mu R - \mu R} |\phi_{R'}| dx \lesssim \sqrt{|R'|} \frac{\log \mu}{\mu}.$$

We use this estimate for all $R' \in \mathcal{S}(R, j, \ell)$, with $j = 1, 2$ and $\ell \leq \sqrt{\mu}$. As there are 2^ℓ member of $\mathcal{S}(R, j, \ell)$, we see that

$$\sum_{j=1}^2 \sum_{\ell=1}^{\sqrt{\mu}} \sum_{R' \subset \mathcal{S}(R, j, \ell)} |\langle \phi_{R'}, f \rangle|^2 \lesssim \frac{|R|}{\sqrt{\mu}}.$$

It remains to consider the case of $\ell \geq \sqrt{\mu}$. Let us consider $j = 1$, the case of $j = 2$ being simmilar. Without loss of generality, we can assume that R is centered at the origin. The difficulty arises from the poor decay of the functions $\phi_{R'}$ for $R' \in \mathcal{R}(k)$ in the direction $(-1, a_k)$. However, these directions are now localized in the direction $(-1, 0)$ since $\ell \geq \sqrt{\mu}$. [By assumption $a_k \simeq 2^{-k}$ and $\ell \geq k$.] Thus we break up the support of f in this way.

$$V_1 := \{(-2\mu|r_1|, -\mu|r_1|) \cup (\mu|r_1|, 2\mu|r_1|)\} \times (-\sqrt{\mu}|r_2|, \sqrt{\mu}|r_2|), \\ V_2 := 2\mu R - \mu R - V_1.$$

The critical inequality is that if g is supported on V_1 , then for all $\ell \geq \sqrt{\mu}$,

$$(4.7) \quad \sum_{R' \in \mathcal{S}(R, 1, \ell)} |\langle \phi_{R'}, g \rangle|^2 \lesssim \mu^{-9/10} \|g\|_2^2.$$

Indeed, from (4.6), we see the inequalities

$$\|\phi_{R'}\|_{L^2(V_1)} \lesssim \mu^{-1}, \quad R' \in \mathcal{S}(R, 1, \ell), \\ \int_{V_1} |\phi_{R'} \phi_{R''}| dx \lesssim \left[\frac{\text{dist}(R', R'')}{2^{-\ell}|r_2|} \right]^{-100}, \quad R', R'' \in \mathcal{S}(R, 1, \ell).$$

The estimate (4.7) is a direct consequence of these two observations. Applying this to a bounded function f supported on V_1 , we see that $\|f\|_2 \leq \|f\|_\infty \sqrt{|V_1|} \lesssim \mu^{3/4}|R|$. This then is consistent with our claim (4.5).

The remaining case concerns bounded functions f supported on V_2 . But in this case, the decay of the functions $\phi_{R'}$ is much better. From (4.6) and the definition of V_2 we see that

$$\int |\phi_{R'}| dx \lesssim \sqrt{R'} 2^{-10\ell}, \quad R' \in \mathcal{S}(R, 1, \ell).$$

This is more than enough to conclude (4.5) for bounded functions f supported on V_2 , finishing the proof of that inequality.

A Appendix: Carleson Measures

We include a proof of some results related to the BMO theory of the bidisk cited above. Our results will be slightly more general than we need. Let us begin with the John–Nirenberg Lemma. For the collection of dyadic rectangles \mathcal{R} in the plane, let $a : \mathcal{R} \rightarrow \mathbb{R}_+$ be a map. Define

$$\|a\|_{CM,p} := \sup_{U \subset \mathbb{R}^2} |U|^{-1} \left\| \sum_{R \subset U} a(R) \mathbf{1}_R \right\|_p, \quad 0 \leq p < \infty,$$

where the supremum is over all open sets $U \subset \mathbb{R}^2$. This is a possible definition of the norm of a Carleson measure. The John–Nirenberg inequality asserts that all of these possible definitions are equivalent, up to constants.

A.1. Lemma. *For all $0 \leq p, q < \infty$, we have*

$$\|a\|_{CM,p} \lesssim \|a\|_{CM,q}$$

Proof. For open sets $U \subset \mathbb{R}^2$ define

$$F_U(x) := \sum_{R \subset U} a(R) \mathbf{1}_R(x).$$

It suffices to prove the inequality above for a restricted range of p and q . We begin with the case of $\|a\|_{CM,1} \lesssim \|a\|_{CM,p}$, for some $0 < p < 2$. It suffices to fix a choice of a with $\|a\|_{CM,p} \leq 1$, and $\text{supp}(F_{\mathcal{R}}) = U$ has finite measure. We need only show that

$$\int F_U dx \lesssim |U|$$

And to this end, it suffices to demonstrate that there is an open set $V \subset U$, with $|V| < |U|/2$, for which

$$\int F_U \lesssim |U| + \int F_V dx.$$

It is clear that this inequality can then be inductively applied to V to yield the proof of the desired inequality.

We define V as follows. For some constant $0 < \epsilon < 1/2$, set $E := \{F_U > \epsilon^{-2/p}\}$, and $V := \{M \mathbf{1}_E > \epsilon\}$. By the boundedness of the strong maximal function, for ϵ appropriately small, the measure of V is at most one-half the measure of U .

But at the same time, if R is a dyadic rectangle with $R \not\subset V$, then $|R \cap E| < |R|/2$. Hence,

$$\begin{aligned} \int (F_U - F_V) dx &= \sum_{\substack{R \subset U \\ R \not\subset V}} a(R) |R| \\ &\lesssim \sum_{\substack{R \subset U \\ R \not\subset V}} a(R) |R \cap E^c| \\ &\lesssim \int_{E^c} (F_U) dx \end{aligned}$$

$$\begin{aligned} &\lesssim \int F_U^p dx \\ &\lesssim |U|. \end{aligned}$$

This follows since we have an upper bound of F_U off of the set E . This case has been proved.

We now turn to the estimate $\|a\|_{CM,p} \lesssim \|a\|_{CM,1}$, for choices of $1 < p < \infty$. This is all that remains to be done. Indeed this is the case that is explicitly proved in [1], but we include the details for the convenience of the reader.

Due to the recursive nature of the definition of energy, it suffices to prove the following. Fix a choice of a with $\|a\|_{CM,p} \leq 1$, and $\text{supp}(F_{\mathcal{R}}) = U$ has finite measure. Then, there is an open set $V \subset U$, with $|V| < |U|/4$, for which

$$\|F_U\|_p \lesssim |U|^{1/p} + \|G_V\|_p.$$

This is done by way of duality. Thus let p' be the conjugate index to p and select a non-negative $h \in L^{p'}$ of norm one so that $\langle F_U, h \rangle = \|F_U\|_p$. The open set V is then

$$V := \bigcup_{R \in \mathcal{R}} \{R : \frac{1}{|R|} \int_R h dx > \lambda\},$$

where we choose $\lambda > 0$ momentarily.

By the boundedness of the strong maximal function,

$$|V| \leq C_r \lambda^{-p'} \|h\|_{p'}^{p'} = \frac{1}{2} |U|$$

if we take $\lambda \simeq |U|^{1/p'}$. But then

$$\begin{aligned} \|F_U\|_p &= \langle F_U, h \rangle \leq \|F_V\|_p + \sum_{\substack{R \in \mathcal{R} \\ R \not\subset V}} |a_s| \int_R h dx \\ &\leq \|F_V\|_p + \lambda \sum_{s \in \mathbf{T}} |a_s| \\ &\leq \|F_V\|_p + \lambda |U|, \end{aligned}$$

which proves our inequality by the choice of λ . □

The second topic is that of Journé's Lemma, which in any of its various forms must be stated in terms of these quantities. Fix an open set U and a rectangle $R \subset U$. Then

$$\mu_R := \sup\{\mu : \mu R \subset \{M\mathbf{1}_U > 1/2\}\}.$$

Specifically, in this paper we assumed this lemma. [For a more precise result, see [6].]

A.2. Lemma. Fix $\epsilon > 0$. Let $a : \mathcal{R} \rightarrow \mathbb{R}_+$ be such that for all open sets U and all dyadic $R \subset U$,

$$\sum_{R' \subset R} a_{R'} \lesssim \mu^{-\epsilon} |R|.$$

Then $\|a\|_{CM,1} \lesssim 1$.

Proof. Notice that if \mathcal{R}' is a collection of rectangles for which

$$|R \cap R'| < \frac{1}{2}(|R| \wedge |R'|), \quad R, R' \in \mathcal{R}'$$

then

$$\sum_{R \in \mathcal{R}'} |R| \leq 2|\cup\{R : R \in \mathcal{R}'\}|.$$

Our objective is to arrange the collection of rectangles into subcollections which are “nearly disjoint” in this sense, and for which μ_R is approximately the same energy.

For integers $k \geq 0$, let \mathcal{R}_k be those dyadic rectangles which satisfy (1) $R \subset U$, (2) R is maximal among all rectangles satisfying (1), (3) and $2^k \leq \mu_R < 2^{k+1}$. Then let \mathcal{R}'_k be a subcollection of \mathcal{R} in which the lengths of the two sides of R are restricted to be in $2(k+1)\mathbb{Z}+j$ for the first side and $2(k+1)\mathbb{Z}+j'$, with $0 \leq j, j' < 2(k+1)$. Certainly there are at most $4(k+1)^2$ such subcollections \mathcal{R}'_k . But, by maximality, these collections are “nearly disjoint” in the sense of the previous paragraph. Hence,

$$\sum_{R \in \mathcal{R}'_k} \sum_{R' \subset R} a_{R'} \leq 2^{-\epsilon k} \sum_{R \in \mathcal{R}'_k} |R| \leq 2^{-\epsilon k} |U|.$$

This estimate is summable over the $4(k+1)^2$ possible choices of \mathcal{R}'_k and over $k \geq 0$. □

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